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On the duality arising from the Class II subgroups of the Infinite Dimensional Rotation Group.

飛田武幸
名古屋大学、名城大学　名誉教授

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概 要

ブラウン運動もホワイトノイズも、よく知られていると思われているが、実際はまだ解明しなければならないことや未開発の分野が残されている。それらを調べるための一環として、ここでは次の２点をとりあげたい。

1)　無限次元回転群はホワイトノイズ測度を特徴づけるので、その研究はホワイトノイズ解析に直結する。この群の新たな部分群または部分半群を探しだし、調和解析につなげることを試みる。半群はいわゆるクラス II に属するが、そこには新たな Duality がみられる。

同じく回転群に関連して派生する話題として次のことを扱う。

2) クラス I と クラス II の部分群の特性と関係について、
前者は有限元回転の極限（広い意味で）としてみられるが、後者は本質的に連続無限的で、前者から後者への移行には注意すべきことが多い。いわば Digital から Analogue への移行である。ランダムな場合であり、analogue では変数として $\dot{B}(t)$ を用いる。それは長さ無限大のベクトルであり、その非線形関数の構成には「繰り込み」が必要であり、微分作用素の digital のときとは異質である。その解析的、かつ確率論的特長に注目したい。また量子ダイナミックスとの関連にも注意したい。関連する事項の詳細は別稿で扱う。

1 Introduction

We are in search of profound properties of Brownian motion $B(t)$ and white noise $\dot{B}(t)$ that remain not yet so much investigated. Indeed, there are many such subjects. Among others, *invariance* of the probability distribution μ of white noise is now attractive for us. For the study of the invariance in question, we can provide a powerful tool from the theory of transformation group. That is the infinite dimensional rotation group, in particular, the so-called class II subgroup. In the course of this study, we have found a characteristic property in line with *duality.* Further, we have found some good connections with quantum dynamics; e.g. conformal invariance of quantum fields.

2 Infinite dimensional rotation group and the group $Diff_{+}(S^1)/Rot(S^1)$

We start with the general definition of the infinite dimensional rotation group after H. Yoshizawa 1969.

Take a nuclear space E and let $O(E)$ be a collection of memebers g such that

1) g is a linear isomorphism of of E,

2) g is orthogonal:

$$\|g\xi\| = \|\xi\|.$$

For the present purpose we specify E to be D_0 in the sense of Gel'fand:

$$D_0 = \{f : f(u) \in C^\infty, f(1/u)\frac{1}{|u|} \in C^\infty\}. \qquad (2.1)$$

We can establish an isomorphism

$$D_0 \cong D(S^1),$$

where $D(S^1)$ is the space of C^∞-functions on the unit circle S^1. The actual isomorphism may be given by the mapping:

$$\gamma : \xi(\theta) \to f(u) = (\gamma\xi)(u) = \xi(2\tan^{-1} u)\frac{\sqrt{2}}{\sqrt{1+u^2}}. \tag{2.2}$$

The topology of the space D_0 is introduced so as to be isomorphic to $C^\infty(S^1)$. As a result the space D_0 is a σ-Hilbert nuclear space.

To fix the idea, we shall take a nuclear space E to be either D_0 or D_{00} which will be introduced in Section 4.

i) Since $O(E)$ is very big (neither compact nor locally compact), we take subgroups that can be managed. First the entire group is divided into two parts: Class I and Class II.

   The Class I involves members that can be determined by using a base (or coordinate vectors), say $\{\xi_n\}$ of E:

   While any member of the class II should come from a diffeomorphism of the parameter space $\bar{R}$, the one-point compactification of R.

ii) We are interested in new subgroups in the class II that are illustrated below.

As in [9], we can define the class II subgroups of $O(E)$.

Definition 1. Let g be a member of $O(E)$ defined in the form

$$(g\xi)(u) = \xi(f(u))\sqrt{|f'(u)|}, \tag{2.3}$$

where f is a diffeomorphism of $\bar{R}$. If such g belongs to $O(E)$, then, g is said to be in the class II. If a subgroup G of $O(E)$ involves members in class II only, then G is a subgroup of class II.

Proposition 2.1 *A class II subgroup of $O(E)$ is isomorphic to some subgroup G of $Diff_+(S^1)/Rot(S^1)$.*

Proof. Compare the norm, For $E = D_0$ a member of $O(E)$ preserves the L^2-norm, Use the transformation γ. Then, we can see that for $f = \gamma\xi$

$$\int_{-\infty}^{\infty} f(u)^2 du = \int_{-\pi}^{\pi} \xi(\theta)^2 d\theta$$

holds. Hence a member $g \in Diff(S^1)$ corresponds to a member of $O(E)$ only when g preserves the $L^2(S^1)$-norm.

Take a class II subgroup U of $O(E)$ such that

$$U \cong Diff(S^1) \cap V,$$

where V is the unit ball of $L^2(S^1, d\theta)$,

Definition 2.2 *A one parameter subgroup $g_t, t \in R^1$, of U is called a* **whisker** *if it is expressed in the form*

$$(g_t)(\xi)(u) = \xi(\psi_t(u))\sqrt{|\psi_t'(u)|} \tag{2.4}$$

where $\psi_t(u) = f^{-1}(f(u)+t)$, and if g_t is continuous in t.

The collection of whiskers is denoted by W.

Some more details regarding the whiskers shall be discussed in Section 3. The next subject to be reminded is the adjoint, denoted by g^*, of $g \in O(E)$.

The collection $O^*(E^*)$ of the adjoint operators g^* forms a group which is isomorphic to $O(E)$. The significance of the group $O^*(E^*)$ is that every g^* in $O^*(E^*)$ keeps the white noise measure μ to be *invariant*:

$$g^*\mu = \mu. \tag{2.5}$$

From this equality, starts the characterization of μ by using the rotation group.

One might think that $O(E)$ is a limit of the finite dimensional rotation groups $SO(n)$ as $n \to \infty$, but not quite. The limit can occupy a very minor part of $O(E)$: of course it is almost impossible to measure the size of the limit occupied in the entire group $O(E)$. Set

$$G_\infty = ind.lim_n G_n,$$

where $G_n \cong SO(n)$. The G_∞ is in class I.

In what follows we shall discuss particularly subgroups belonging to the class II, in particular W.

3 Whiskers

Subgroups in the Class I have, so far, been rather well- investigated. We shall, therefore, study the Class II,

First we shall have a brief review of the known results so that we can find some hints to find new good subgroups of $O(E)$.

Each member of the class W, say $\{g_t, t : \text{real}\}$, should be defined by a system of parameterized diffeomorphisms $\{\psi_t(u)\}$ of $\bar{R} = R \cup \infty$. Namely, as in (2.4).

We are interested in a subgroup that is consisting of whiskers and that can be made to be a local Lie group embedded in $O(E)$. In what follows the basic nuclear space is specified to D_0 defined before (also see [3]).

More practically, we restrict our attention to the case where $g_t, t \in R$ has the (infinitesimal) generator

$$\alpha = \frac{d}{dt} g_t|_{t=0}$$

Note that, by the assumptions of the group property and continuity, a family $\{\psi_t(u), t \in R\}$ is such that $\psi_t(u)$ is measurable in (t, u) and satisfies

$$\begin{aligned}\psi_t \cdot \psi_t &= \psi_{t+s}\\ \psi_0(u) &= u.\end{aligned}$$

Following J. Aczél [1], we have an expression for $\psi_t(u)$:

$$\psi_t(u) = f(f^{-1}(u) + t) \tag{3.1}$$

where f is continuous and strictly monotone. Its (infinitesimal) generator α, if f is differentiable, can now be expressed in the form

$$\alpha = a(u)\frac{d}{du} + \frac{1}{2}a'(u), \tag{3.2}$$

where

$$a(u) = f'(f^{-1}(u)). \tag{3.3}$$

See e.g.[3], [4].

We have already established the results that there exists a three dimensional subgroup of class II with significant probabilistic meanings. The group consists of three whiskers, the generators of which are expressed by $a(u) = 1, a(u) = u, a(u) = u^2$, respectively.

Namely, we show a list:

$$\begin{aligned}s &= \frac{d}{du},\\ \tau &= u\frac{d}{du} + \frac{1}{2},\\ \kappa &= u^2\frac{d}{du} + u\end{aligned}$$

One of the interesting interpretations may be said that they are put together to describe the *projective invariance* of Brownian motion.

Those generators form a base of a three dimensional Lie algebra under the Lie product.

The algebra given above is isomorphic to $sl(2, R)$. This fact can easily be seen by the commutation relations:

$$[\tau, s] = -s$$
$$[\tau, \kappa] = \kappa$$
$$[\kappa, s] = 2\tau$$

There is a remark that the shift with generator s is sitting as a *key member* of the generators. It corresponds to the *flow of Brownian motion*, significance of which is quite clear.

Also, one can take τ to be another key generator. The τ describes the Ornstein-Uhlenbeck Brownian motion which is stationary Gaussian and simple Markov.

An interesting remark is that the above three generators span a vector space isomorphic to $so(2, 1)$.

Let g_t^* be the adjoint operator to g_t. Then, the system $\{g_t^*\}$ again forms a one-parameter group of μ (the white noise measure) preserving transformations g_t^*. The system is a *flow* on the white noise space (E^*, μ).

We are now in a position to have general relationships among the generators of the form (3.2) with the expression (3.3).

Introduce the notation

$$\alpha_a = a(u)\frac{d}{du} + \frac{1}{2}a'(u),$$

where $a(u)$ is assumed to be $\mathbf{C}_1$ class. The collection of such α_a is denoted by $\mathbf{D}$. Then, we have

Triviality It holds that, with the notation $\{a,b\} = ab' - a'b$, for any α_a and α_b

$$[\alpha_a, \alpha_b] = \alpha_{\{a,b\}},$$

$$[\alpha_a, \alpha_b] = -[\alpha_b, \alpha_a],$$

where $[\cdot,\cdot]$ is the Lie product.

Proposition 3.1 *The collection* $\mathbf{D}$ *forms a base of a Lie algebra, which is denoted by* $\mathbf{A}$. *There is no identity.*

It is interesting to find a subalgebra which is expected to have some interesting probabilistic property. Setting $\alpha^p = \alpha_{u^{p-1}}, p \in Z$, in particular, we shall discuss some more details in the next section.

4 Half whiskers

The results of this section mostly come from [13].

We are now in search of *new* whiskers that show some significant probabilistic properties hopefully like the three whiskers in the last section under somewhat general setup. There a whisker may be changed to a half-whisker under mild restrictions.

First we recall the notes [11] p. 60, Section O_∞ 1, where a new class of whiskers has been proposed, in reality, most of the members are half whiskers. Let us repeat the proposal.

$$\alpha^p = u^{p+1}\frac{d}{du} + \frac{p+1}{2}u^p, \ u \geq 0, \tag{4.1}$$

is suggested to be investigated, where p is not necessarily be integer. (The power $p+1$ was written as α

in [11], but to avoid notational confusion, we write p instead of α.)

Since fractional power p is involved, we tacitly assume that u is non-negative, We, therefore,take a white noise with time-parameter $[0, \infty)$. The basic nuclear space E is chosen to be D_{00} which is isomorphic to D_0, eventually isomorphic to $C^\infty(S^1)$.

We are now ready to state a partial answer.

As was remarked in the last section, the power 1 is the key number and, in fact, it is exceptional. In this case the variable u runs through R, that is, corresponds to a whisker with generator τ. In what follows, we *exclude* the case $p = 0$.

We remind the relationship between f and $a(u)$ that appears in the expressions of $\psi_t(u)$ and α, respectively. The related formulas are the same as in the case where u runs through R.

Assuming differentiability of f we have the formula (3.8). For $a(u) = u^p$, the corresponding $f(u)$ is determined. Namely,

$$u^p = f'(f^{-1}(u)).$$

An additional requirement for f is concerned with the domain of f, namely f should be a map from the entire $[0.\infty)$ onto itself. Hence, we have

$$f(u) = c_p u^{\frac{1}{1-p}}, \tag{4.2}$$

where $c_p = (1-p)^{1/(1-p)}$.

We, therefore, have

$$f^{-1}(u) = (1-p)^{-1} u^{1-p}. \tag{4.3}$$

We are ready to define a transformation g_t^p acting on D_{00} by

$$(g_t\xi)(u) = \xi(c_p(\frac{u^{1-p}}{1-p}+t)^{1/(1-p)})\sqrt{\frac{c_p}{1-p}(\frac{u^{1-p}}{1-p}+t)^{p/(1-p)}u^{-p}}. \tag{4.4}$$

Note that f is always positive and maps $(0, \infty)$ onto itself in the ordinary order in the case $p < 1$; while in the case $p > 1$ the mapping is in the reciprocal order .

The exceptional case $p = 1$ is refered to the literature [4]. It has been well defined.

Then, we claim, still assuming $p \neq 1$, the following theorem.

Theorem 4.1 *i) g_t^p is a member of $O(D_{00})$ for every $t > 0$.*

ii) The collection $\{g_t^p, t \geq 0\}$ forms a continuous semi-group with the product $g_t^p \cdot g_s^p = g_{t+s}^p$ for $t, s \geq 0$.

iii) The generator of g_t^p is α^p given by (4.1) up to constant.

Proof. Assertion i) comes from the structure of D_{00}.

Assertions ii) and iii) can be proved by actual elementary computations.

Definition A continuous semi-group $g_t, t \geq 0$, each member of which comes from $\psi_t(u)$ is called a *half whisker.*

Theorem 4.2 *The collection of half whiskers $g_t^p, t \geq 0, p \in R$, generates a local Lie semi-group G_L:*

$$G_L = \text{generated by } \{g_{t_1}^{p_1} \cdots g_{t_n}^{p_n}\}$$

The definition of a local Lie group is found, e.g. in W. Miller, Jr. [11]. A semi-group is defined similarly.

5 Lie algebra and duality

The collection $\{\alpha^p; p \in R\}$ generates a vector space $\mathbf{g}_L$, where the Lie product $[\cdot,\cdot]$ is introduced.

Proposition 5.1 *The vector space $\mathbf{g}_L$ forms a Lie algebra with the usual Lie prodect.*

Note that the exceptional power 1 is now included. With this understanding, we have

Theorem 5.2 *The space $\mathbf{g}_L$ is a Lie algebra parameterized by $p \in R$. It is associated with the local Lie semi-group G_L.*

Proof. We have

$$[\alpha^p, \alpha^q] = (q-p)u^{p+q+1}\frac{d}{du} + \frac{1}{2}(q-p)(p+q-1)u^{p+q}. \tag{5.1}$$

The result is $(q-p)\alpha^{p+q}$. This proves the theorem.

In fact, we have an infinite dimensional Lie algebra, the base of which consists of one-parameter system of generators of half whiskers.

[Note] If $\mathbf{g}_L$ is slightly modified to $\mathbf{g}'_L = \{\frac{1}{p}\alpha^p\}$, then the exceptional member $\alpha^0 (= \tau)$ plays the role of the identity:

$$[\alpha^p, \alpha^0] = \alpha^p.$$

In addition, α^0 can be the generator of a whisker and it plays really a central role.

Duality.

With respect to α^0 we can see a duality

$$\alpha^p \iff \alpha^{-p}$$

For every p, the $(g_t^p)^*$ is a semigroup of μ-measure preserving transformations. We may, therefore, define a Gaussian process $X^p(t)$ in such a manner that

$$X^p(t) = \langle (g_t^p)^* x, \xi \rangle,$$

where $x \in E^*(\mu)$.

We have much freedom to choose ξ, in fact, we may choose the indicator function $\chi_{[0,1]}(u)$.

By a simple computation we can see that for $p < 1, 0 < h < 1$

$$E(X^p(t+h), X^p(t)) = \gamma(h),$$

holds, that is a function only of h.

6 Concluding remarks.

1. We shall propose a more general theory, where it is possible to propose many kinds of half whiskers. Namely, we may consider general infinitesimal generators, where the functions $a(u)$ in (3.8) or f in (3.6) are restricted so as to define subgroups of $O(D_{00})$.

2. As is easily seen, the algebra $\mathbf{g}_L$ is *perfect.* that is

$$[\mathbf{g}_L, \mathbf{g}_L] = \mathbf{g}_L,$$

that is, the derived algebra coincides with itself. Hence there exists the universal central extension. It is our hope that we can follow the line of studying the Virasoro algebra.

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